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Introduction to ''High Resolution Schemes for Hyperbolic Conservation Laws''

This paper was a landmark; it introduced a new design principle—total variation diminishing schemes—that led, in Harten's hands, and subsequently in the hands of others, to an efficient, robust, highly accurate class of schemes for shock capturing free of oscillations. The citation index lists 429 references to it, not only in journals of numerical analysis and computational fluid dynamics, but also in journals devoted to mechanical engineering, astronautics, astrophysics, geophysics, nuclear science and technology, spacecraft and rockets, plasma physics, sound and vibration, aerothermodynamics, hydraulics, turbo and jet engines, and computer vision and imaging.

The basic idea is a simple condition for a three-point explicit scheme for a scalar quantity in one space variable to be variation diminishing, a condition that has been extended to n point schemes in [1] and [2]. The next step is the construction of a second-order five-point scheme that, when interpreted as a three-point scheme, is variation diminishing. Even when the equation in question is linear, the scheme proposed is nonlinear, a surprising idea though one already present in Harten's master's and doctoral dissertations. It is precisely this nonlinearity that allows the escape from the class of monotone schemes, which are necessarily of first-order accuracy.

Harten discusses the importance of entropy conditions for picking out the physically relevant solutions. He introduces an entropy fix in the choice of the viscosity term that prevents violation of the entropy condition.

Harten constructs variation diminishing schemes for systems of conservation laws, making use of a notion of total variation that was introduced by Glimm. He also employs Roe-type averages. The argument here is somewhat formal, but its soundness is born out by impressive numerical experiments for the Euler equations of gas dynamics.

Inspired by Harten's paper, van Leer, Osher, Roe, and other researchers used the variation diminishing principle to construct and analyze new difference schemes.

In the last section the scheme is extended to two-dimensional problems, using Strang-type dimensional splitting to achieve second-order accuracy, and is used to calculate the flow of air through a partially obstructed duct. The main features of such a flow are recognizable in the numerical results presented. Subsequently, much effort was devoted by others to constructing variation-diminishing schemes for the calculation of two-dimensional flows. Such efforts are of course doomed to failure, since the total variation of exact solutions can blow up through the mechanism of focusing. Even in the scalar case, where no focusing takes place, Goodman and Leveque [4] showed that there are no TVD schemes. The only viable two-dimensional analogue of variation-diminishing is energy (or entropy) diminishing; it was shown in [3] how to combine some ideas of Friedrichs with those of Harten to construct such schemes. The arguments are somewhat formal, but their soundness is demonstrated by numerical experiments.

The review paper [5] gives an overview of the shock capturing scheme in general and TVD schemes in particular.

Harten originally called his schemes variation diminishing, abbreviated TVD; when Osher pointed out the usual meaning of these initials, the name was switched to total variation nonincreasing (TVNI), but was eventually settled on the more euphonious TVD.

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